JOURNAL OF APPROXIMATION THEORY 32, 292–305 (1981)

Precise Orders of Strong Unicity Constants for a Class of Rational Functions

MYRON S. HENRY*

Department of Mathematics, Central Michigan University, Mount Pleasant, Michigan 48859

AND

JOHN J. SWETITS

Department of Mathmatical Sciences, Old Dominion University, Norfolk, Virginia 23508

Communicated by E. W. Cheney

Received March 14, 1980

Let $\hat{\mathbf{R}} \subseteq C[-1, 1]$ denote a certain class of rational functions. For each $f \in \hat{\mathbf{R}}$, consider the polynomial of degree at most n that best approximates f in the uniform norm. The corresponding strong unicity constant is denoted by $M_n(f)$. Then there exist positive constants α and β, not depending on n, such that $\alpha n \leqslant M_n(f) \leqslant \beta n$, $n = 1, 2, \ldots$.

1. INTRODUCTION

Let $C(I)$ denote the space of real valued, continuous functions on the interval $I = [-1, 1]$, and let $\Pi_n \subseteq C(I)$ be the space of real polynomials of degree at most n. Denote the uniform norm on $C(I)$ by $\|\cdot\|$. For each $f \in C(I)$ with best approximation $B_n(f)$ from Π_n, there is a smallest constant $M_n(f) > 0$ such that for any $p \in \Pi_n$,

$$\|p - B_n(f)\| \leqslant M_n(f)(\|f - p\| - \|f - B_n(f)\|). \tag{1.1}$$

Inequality (1.1) is the well-known strong unicity theorem [3], and hereafter $M_n(f)$ is defined to be the strong unicity constant.

The behavior of the sequence

$$\{M_n(f)\}_{n=0}^{\infty} \tag{1.2}$$

* Part of the research for this paper was done as a visiting professor at Old Dominion University, August 1979–July 1980.

0021-9045/81/080292-14$02.00/0

has been the subject of several recent papers. (In addition to the references of the current paper, the interested reader is referred to a recent survey paper by Bartelt and Schmidt [1] and to the bibliographies of [5, 7].) In [5], Henry and Huff introduced the terminology "precise order of $M_n(f)$." Definition 1 below is a modification of the definition appearing in [5].

DEFINITION 1. Let $f \in C(I)$, and suppose there exist positive constants α and β, a natural number N, and a positive real valued function c with domain the natural numbers satisfying

$$\alpha c(n) \leqslant M_n(f) \leqslant \beta c(n), \qquad \text{for all } n \geqslant N. \tag{1.3}$$

Then $M_n(f)$ is said to be of precise order $c(n)$.

For certain functions $f \in C(I)$, the authors [7], in collaboration with S. E. Weinstein, have shown that

$$\alpha n \leqslant M_n(f) \leqslant \beta n^2. \tag{1.4}$$

Although an entire class of non-polynomial functions that satisfy (1.4) is given in [7], to date the only non-polynomial function for which the precise order of $M_n(f)$ has been established [5] is $f(x) = 1/(x - \lambda)$, $\lambda \geqslant 2$, $x \in I$.

The goal of the present paper is to establish the precise order of $M_n(f)$ for every $f \in \hat{\mathbf{R}}$, where $\hat{\mathbf{R}}$ is a certain class of rational functions [9].

2. PRELIMINARIES

For $f \in C(I)$, $e_n(f)(x) = f(x) - B_n(f)(x)$. Let

$$E_n(f) = \{x \in I : |e_n(f)(x)| = \|e_n(f)\|\} \tag{2.1}$$

be the set of extreme points of the error curve $e_n(f)$. Suppose that $\mathscr{A} = \{x_0, x_1, ..., x_{n+1}\} \subseteq E_n(f)$ is an alternate [3, p. 75] for $e_n(f)$. Define $q_{in} \in \Pi_n$, $i = 0, ..., n+1$, by

$$\begin{gathered} q_{in}(x_j) = \operatorname{sgn} e_n(f)(x_j), \\ j = 0, ..., n+1, \quad j \neq i; \quad i = 0, ..., n+1. \end{gathered} \tag{2.2}$$

If $E_n(f)$ consists of precisely $n+2$ points $x_0 < x_1 < \cdots < x_{n+1}$, then Henry and Roulier [6], utilizing the work of Cline [4] have shown that

$$M_n(f) = \max_{0 \leqslant j \leqslant n+1} \{\|q_{in}\|\}. \tag{2.3}$$

Hereafter let

$$\gamma_n(x) = \operatorname{sgn} e_n(f)(x), \qquad x \in I. \tag{2.4}$$

The following theorem is fundamental to the subsequent analysis.

THEOREM 1 (Schmidt [10]). *Let $q \in \Pi_n$ satisfy $\gamma_n(x)\, q(x) \leqslant 1$ for all $x \in E_n(f)$ and $\|q\| = M_n(f)$. Let $A = \{x \in E_n(f)\colon \gamma_n(x)\, q(x) = 1\}$, and select $x^* \in I$ such that $|q(x^*)| = \|q\|$. Then there exist $n+1$ points $y_0 < y_1 < \cdots < y_n$ in A such that either*

(i) *$x^* < y_0$ and $-\operatorname{sgn} q(x^*)$, $\gamma_n(y_0), \gamma_n(y_1),..., \gamma_n(y_n)$ alternate in sign;*

or

(ii) *$x^* > y_n$ and $\gamma_n(y_0), \gamma_n(y_1),..., \gamma_n(y_n)$, $-\operatorname{sgn} q(x^*)$ alternate in sign;*

or

(iii) *$y_{i-1} < x^* < y_i$ for some $i = 1,..., n$, and $\gamma_n(y_0),..., \gamma_n(y_{i-1})$, $-\operatorname{sgn} q(x^*)$, $\gamma_n(y_i),..., \gamma_n(y_n)$ alternate in sign.*

The class of rational functions for which precise orders of strong unicity constants will be established is now described. These ratonal functions are extensively analyzed by Rivlin [9].

Let a and b be non-negative integers with $a > 0$. If $|t| < 1$, define $f \in C[-1, 1]$ by

$$f(x) = \sum_{j=0}^{\infty} t^j T_{aj+b}(x), \tag{2.5}$$

where T_k is the kth degree Chebyshev polynomial. Then Rivlin [9] shows that

$$f(x) = \frac{T_b(x) - tT_{|b-a|}(x)}{1 + t^2 - 2tT_a(x)}. \tag{2.6}$$

Let $\mathbf{R} \subseteq C[-1, 1]$ be the set of all rational functions defined by (2.5) and (2.6). If α and β are real numbers, define $\hat{\mathbf{R}}$ by

$$\hat{\mathbf{R}} = \{ \in C[-1, 1]\colon h = \alpha f + \beta, f \in \mathbf{R}\}. \tag{2.7}$$

Section 3 is devoted to showing that if $u \in \hat{\mathbf{R}}$, then $M_n(u)$ is of precise order n. Since $M_n(\alpha f + \beta) = M_n(f)$, it will be sufficient to show that $M_n(f)$ is of precise order n for every $f \in \mathbf{R}$.

Rivlin [9] establishes for $f \in \mathbf{R}$ that

$$B_{ak+b}(f)(x) = \sum_{l=0}^{k} t^l T_{al+b}(x) + \frac{t^{k+2}}{1 - t^2}\, T_{ak+b}(x), \qquad x \in I$$

and that $B_{ak+b}(f) = B_j(f)$ for $j = ak+b, \ldots, a(k+1)+b-1$. Let $n_k = ak+b$, $k = 0, 1, \ldots$. Then

$$B_{n_k}(f) = B_j(f) \qquad \text{for } \; j = n_k, n_k+1, \ldots, n_{k+1}-1. \tag{2.8}$$

Furthermore, with $x = \cos\theta$,

$$e_j(f)(x) = \frac{t^{k+1}}{1-t^2} \frac{A(\theta)}{B(\theta)}, \qquad j = n_k, n_k+1, \ldots, n_{k+1}-1, \tag{2.9}$$

where

$$\frac{A(\theta)}{B(\theta)} = \cos[n_k\theta + \phi], \tag{2.10}$$

and where

$$\cos\phi = \frac{-2t + (1+t^2)\cos a\theta}{1+t^2-2t\cos a\theta}, \qquad \sin\phi = \frac{(1-t^2)\sin a\theta}{1+t^2-2t\cos a\theta}. \tag{2.11}$$

In [9] it is noted that $A(\theta)/B(\theta) = \pm 1$ alternately at $n_{k+1}+1$ points on $[0, \pi]$, which by (2.9) is precisely what is needed to insure (2.8).

It should be noted that (2.9) implies

$$E_{n_k}(f) = E_{n_k+1}(f) = \cdots = E_{n_{k+1}-1}(f). \tag{2.12}$$

Thus for $E_j(f)$, $j = n_k, n_k+1, \ldots, n_{k+1}-2$, the cardinality $|E_j(f)|$ of $E_j(f)$ exceeds $j+2$. However, for $j = n_{k+1}-1$, $k = 0, 1, \ldots$,

$$|E_{n_{k+1}-1}(f)| = n_{k+1}+1. \tag{2.13}$$

For $a = 1$, $b = 0$, and proper choices of α, β, and t, $f(x) = 1/(x-\lambda)$ is included in $\hat{\mathbf{R}}$. However, in the context of the present paper, $a > 0$ and b may be any non-negative integers, and consequently $|E_j(f)|$ may exceed $j+2$ infinitely often. The analysis given in [5] depended on $|E_j(f)| = j+2$ for $j = 0, 1, \ldots$. Thus the arguments below are necessarily more complex than those of [5].

3. Precise Orders for Certain Rational Functions

In this section the principle result of the present paper is established: for every $f \in \mathbf{R}$, $M_n(f)$ is of precise order n. The proof of this assertion will be accomplished through a series of lemmas, culminating in Theorems 2 and 3.

LEMMA 1. *For $f \in \mathbf{R}$, the extreme points of $E_j(f)$ consist of* -1, $+1$, *and the* $n_{k+1}-1$ *roots of*

$$g(x)=aT'_{n_k}(x)[-2t+(1+t^2)\,T_a(x)]+n_kT_{n_k}(x)(1-t^2)\,T'_a(x),$$
$$j=n_k, n_k+1,..., n_{k+1}-1. \tag{3.1}$$

Proof. Equation (2.10) and the remarks below (2.11) imply that there exist $n_{k+1}+1$ values $n_k\theta_i+\phi_i$ such that $A(\theta_i)/B(\theta_i)$ alternates between -1 and $+1$ for $i=0,..., n_{k+1}$. Thus

$$\sin(n_k\theta_i+\phi_i)=0, \qquad i=0,..., n_{k+1}, \tag{3.2}$$

but

$$\sin(n_k\theta+\phi)=\sin n_k\theta\cos\phi+\cos n_k\theta\sin\phi.$$

Substituting (2.11) into this equation and utilizing $T_l(x)=\cos l\theta$, $x=\cos\theta$, yields

$$\begin{aligned}\sin(n_k\theta+\phi)=\sin\theta/[n_ka(1+t^2-2t\cos a\theta)]&\\ \times\{aT'_{n_k}(x)[-2t+(1+t^2)\,T_a(x)]&\\ +n_kT_{n_k}(x)(1-t^2)\,T'_a(x)\}.&\end{aligned} \tag{3.3}$$

Since $|t|<1$, Eqs. (3.2) and (3.3) imply the conclusion. ∎

LEMMA 2. *For $f \in \mathbf{R}$,*

$$\begin{aligned}&[e_j(f)(x)/\|e_j(f)\|][1+t^2-2tT_a(x)]\\ &\quad=T_{n_k}(x)[-2t+(1+t^2)\,T_a(x)]-[(1-x^2)/n_ka]\,T'_{n_k}(x)\,T'_a(x)(1-t^2),\\ &\qquad j=n_k, n_k+1,..., n_{k+1}-1.\end{aligned} \tag{3.4}$$

Proof. Equations (2.9) and (2.10) imply that

$$e_j(f)(x)/\|e_j(f)\|=\cos(n_k\theta+\phi), \qquad j=n_k, n_k+1,..., n_{k+1}-1.$$

Equation (3.4) is obtained by utilizing (2.11), $x=\cos\theta$, $T_l(x)=\cos l\theta$, and algebraic manipulation. ∎

Define $\bar{Q}_{m+1}$ by

$$\begin{aligned}\bar{Q}_{m+1}(x)=T_{n_k}(x)[-2t+(1+t^2)\,T_a(x)]&\\ -[(1-x^2)/n_ka]\,T'_{n_k}(x)\,T'_a(x)(1-t^2),&\end{aligned} \tag{3.5}$$

where $m = n_{k+1} - 1$. Equality (2.13) implies that $|E_m(f)| = m+2$. From Lemma 1, -1, $+1$, and the zeros of (3.1) constitute the elements of $E_m(f)$; label these as

$$-1 = x_0 < x_1 < \cdots < x_{n_{k+1}} = 1. \tag{3.6}$$

For each $k = 0, 1,..., m+1$, define $Q_{m+1} \in \Pi_{m+1}$ by

$$Q_{m+1}(x_k) = \gamma_m(x_k), \tag{3.7}$$

where γ_m is given by (2.4). Let $q_{im} \varepsilon \Pi_m$ be determined by

$$q_{im}(x_k) = \gamma_m(x_k), \qquad k = 0,..., m+1, \quad k \neq i. \tag{3.8}$$

Since $|E_m(f)| = m+2$, (2.3) implies that

$$M_m(f) = \max_{0 \leqslant i \leqslant m+1} \|q_{im}\|. \tag{3.9}$$

Thus initially $\|q_{im}\|$, $i = 0,..., m+1$, is estimated.

LEMMA 3. *Suppose that Q_{m+1} and q_{im}, $i = 0,..., m+1$, are defined by* (3.7) *and* (3.8), *respectively. Let a_{m+1} be the coefficient of x^{m+1} in Q_{m+1}. Then*

$$q_{im}(x) = Q_{m+1}(x) - \frac{a_{m+1}(x^2-1)\, g(x)}{a(m-a+1)\, 2^m (x - x_i)},$$
$$i = 0,..., m+1, m = n_{k+1} - 1. \tag{3.10}$$

Proof. Equations (3.7) and (3.8) imply that

$$q_{im}(x) = Q_{m+1}(x) - a_{m+1} \prod_{\substack{k=0 \\ k \neq i}}^{m+1} (x - x_k). \tag{3.11}$$

From Lemma 1,

$$w(x) = \prod_{k=0}^{m+1} (x - x_k)$$
$$= \frac{(x^2-1)\, g(x)}{a(m-a+1)\, 2^m}. \tag{3.12}$$

Equations (3.11) and (3.12) now imply the conclusion. ∎

LEMMA 4. *Let g be the polynomial given in* (3.1). *Then*

(a) $\|g\|$ *is of precise order* $(m-a+1)^2$, *and*

(b) $\max_{-1\leqslant x\leqslant 1}|(d/dx)[(x^2-1)\,g(x)]$ *is or precise order* $(m-a+1)^2$.

Proof. For part (a), (3.1) implies that

$$\|g\|\leqslant \alpha\,\|T'_{n_k}\|,$$

where α is some positive constant not depending on n_k. Therefore

$$\|g\|\leqslant \alpha n_k^2=\alpha(m-a+1)^2.$$

On the other hand, (3.1) yields

$$\begin{aligned}\|g\|&\geqslant|g(1)|\\&=an_k^2(t-1)^2+n_k a^2(1-t^2)\\&=(m-a+1)^2\{a(t-1)^2+a^2(1-t^2)/n_k\}.\end{aligned}$$

Since $|t|<1$, this inequality establishes that

$$\|g\|\geqslant\beta(m-a+1)^2.$$

Thus (a) is proven.

To prove part (b), first let $r(x)=(x^2-1)\,g(x)$. Then

$$r'(x)=2xg(x)+(x^2-1)\,g'(x). \tag{3.13}$$

But from (3.1),

$$\begin{aligned}(x^2-1)\,g'(x)=a(x^2-1)\,&T''_{n_k}(x)[-2t+(1+t^2)\,T_a(x)]\\&+a(x^2-1)\,T'_{n_k}(x)(1+t^2)\,T'_a(x)\\&+n_k(x^2-1)\,T'_{n_k}(x)(1-t^2)\,T'_a(x)\\&+n_k(x^2-1)\,T_{n_k}(x)(1-t^2)\,T''_a(x).\end{aligned} \tag{3.14}$$

Now

$$(1-x^2)\,T''_{n_k}(x)=xT'_{n_k}(x)-n_k^2T_{n_k}(x)$$

and

$$(1-x^2)\,T'_{n_k}(x)=n_k[T_{n_k-1}(x)-xT_{n_k}(x)].$$

These last two equalities and (3.14) imply that

$$\max_{-1\leqslant x\leqslant 1}|(x^2-1)\,g'(x)|=O[(m-a+1)^2]. \tag{3.15}$$

Use of (3.13), (3.15), and part (a) implies that

$$\max_{-1 \leqslant x \leqslant 1} |(d/dx)[(x^2 - 1)\, g(x)]| = O[(m - a + 1)^2]. \tag{3.16}$$

To conclude the proof of part (b) let $x = x_i$, where $x_i \varepsilon E_m$ is any extreme point of $e_m(f)$ except 1 or -1. Now (3.3) implies that

$$n_k a \sin\theta(1 + t^2 - 2t\cos a\theta) \sin(n_k\theta + \phi) = (1 - x^2)\, g(x). \tag{3.17}$$

Differentiating (3.17) with respect to x, evaluating at $x = x_i$, and utilizing (3.2) yields

$$\begin{aligned} &a n_k(1 + t^2 - 2t\cos a\theta_i)\cos(n_k\theta_i + \phi_i)(n_k + d\phi/d\theta)|_{\phi=\phi_i} \\ &\quad = (d/dx)[(x^2 - 1)\, g(x)]|_{x=x_i}. \end{aligned} \tag{3.18}$$

Therefore (2.11), (3.18), and the remarks above (3.2) imply that

$$a n_k^2(1 + t^2 - 2t\cos a\theta_i) + a^2 n_k(1 - t^2) = |(d/dx)[(x^2 - 1)\, g(x)]|_{x=x_i}.$$

Thus

$$\begin{aligned} |(d/dx)[(x^2 - 1)\, g(x)]|_{x=x_i} &\geqslant a n_k^2(1 - t)^2 + a^2 n_k(1 - t^2) \\ &= a(m - a + 1)^2(1 - t^2) + a^2(m - a + 1)(1 - t^2). \end{aligned} \tag{3.19}$$

Inequality (3.19) and (3.16) are equivalent to conclusion (b). ∎

LEMMA 5. *Let a_{m+1} be the coefficient of x^{m+1} in Q_{m+1}. Then*

$$2^m/(1 + |t|)^2 \leqslant |a_{m+1}| \leqslant 2^m/(1 - |t|)^2. \tag{3.20}$$

Proof. By (3.7),

$$Q_{m+1}(x) = \sum_{i=0}^{m+1} \gamma_m(x_i) \frac{w(x)}{(x - x_i)\, w'(x_i)} \tag{3.21}$$

where w is defined in (3.12). Since $\gamma_m(x_k)$ is alternately ± 1, $k = 0, \ldots, m+1$,

$$|a_{m+1}| = \sum_{i=0}^{m+1} 1/|w'(x_i)|. \tag{3.22}$$

Let $\bar{a}_{m+1}$ be the coefficient of x^{m+1} in the $\bar{Q}_{m+1}$ defined in (3.5). Comparing (3.4) with (3.5) yields

$$\bar{Q}_{m+1}(x_k) = \gamma_m(x_k)[1 + t^2 - 2tT_a(x_k)], \qquad k = 0, \ldots, m+1.$$

Therefore

$$\bar{Q}_{m+1}(x) = \sum_{i=0}^{m+1} \gamma_n(x_i)[1 + t^2 - 2tT_a(x_i)] \frac{w(x)}{(x - x_i)\, w'(x_i)}.$$

This equality implies that

$$|\bar{a}_{m+1}| = \sum_{i=0}^{m+1} (1 + t^2 - 2tT_a(x_i)) \Big/ \frac{1}{|w'(x_i)|}. \tag{3.23}$$

Equations (3.22) and (3.23) now imply that

$$\min_i (1 + t^2 - 2tT_a(x_i))\, |a_{m+1}| \leqslant |\bar{a}_{m+1}|$$
$$\leqslant \max_i (1 + t^2 - 2tT_a(x_i))\, |a_{m+1}|$$

from which it follows that

$$(1 - |t|)^2\, |a_{m+1}| \leqslant |\bar{a}_{m+1}| \leqslant (1 + |t|)^2\, |a_{m+1}|. \tag{3.24}$$

But (3.5) implies

$$|\bar{a}_{m+1}| = 2^{n_k + a - 1} = 2^m.$$

This equality and (3.24) now imply (3.20). ∎

Lemma 1–5 now facilitate the proofs of Theorem 2 and 3 below.

THEOREM 2. *Let $f \in \mathbf{R}$, where $a > 0$ and b are non-negative integers. Let $n_k = ak + b$, $k = 0, 1,\ldots$. Then $M_{n_{k+1}-1}(f)$ is of precise order $n_{k+1} - 1$.*

Proof. According to Definition 1 it is sufficient to show that there are positive constants α and β independent of k and a natural number K such that

$$\alpha(n_{k+1} - 1) \leqslant M_{n_{k+1}-1}(f) \leqslant \beta(n_{k+1} - 1),$$

for all $k \geqslant K$. Since $|E_m(f)| = m + 2$, $m = n_{k+1} - 1$, (3.9) is valid. Lemma 3 implies that

$$\|q_{im}\| \leqslant \|Q_{m+1}\| + \frac{|a_{m+1}|}{a(m - a + 1)\, 2^m} \max_{-1 \leqslant x \leqslant 1} \frac{|(x^2 - 1)\, g(x)|}{|x - x_i|},$$
$$i = 0,\ldots, m + 1. \tag{3.25}$$

From (3.21)

$$\|Q_{m+1}\| \leqslant |a_{m+1}| \max_{0 \leqslant i \leqslant m+1} \max_{-1 \leqslant x \leqslant 1} \left| \frac{w(x)}{x - x_i} \right|,$$

and consequently (3.11) and (3.25) imply that

$$\|q_{im}\| \leqslant \frac{2\,|a_{m+1}|}{a(m-a+1)\,2^m} \max_{0 \leqslant i \leqslant m+1} \max_{-1 \leqslant x \leqslant 1} \frac{|(x^2-1)\,g(x)|}{|x - x_i|}. \tag{3.26}$$

Applications of Lemma 4 and Lemma 5 to (3.26) now establish that

$$\|q_{im}\| = O(m-a+1), \qquad i = 0,\ldots, m+1. \tag{3.27}$$

On the other hand, from Lemma 3 and (3.6),

$$q_{m+1,m}(1) = Q_{m+1}(1) - \frac{a_{m+1}\,g(1)}{a(m-a+1)\,2^{m-1}}.$$

Therefore

$$|q_{m+1,m}(1)| \geqslant \frac{2\,|g(1)|}{a(1+t)^2(m-a+1)} - 1, \tag{3.28}$$

but (3.1) implies that

$$g(1) = a(m-a+1)^2(1-t)^2 + (m-a+1)\,a^2(1-t^2).$$

This result and (3.28) now establish that

$$\|q_{m+1,m}\| \geqslant \beta^*(m-a+1). \tag{3.29}$$

Inequality (3.29), equality (3.27), and (3.9) now imply the conclusion of Theorem 2. ∎

THEOREM 3. *Let $f \in \mathbf{R}$, where $a > 0$ and b are non-negative integers. Then $M_n(f)$ is of precise order n.*

Proof. Theorem 1 states that $M_m(f)$ has precise order m, where $m = n_{k+1} - 1$. Because of (2.12),

$$M_{n_k}(f) \leqslant M_{n_k+1} \leqslant \cdots \leqslant M_m(f). \tag{3.30}$$

Also

$$n_{k+1} - n_k = a, \qquad k = 0, 1,\ldots. \tag{3.31}$$

The inequalities in (3.30), equality (3.31), and Theorem 1 imply that

$$M_{n_k+j}(f) = O(n_k + j), \qquad j = 0, \dots, a-1. \tag{3.32}$$

Appealing again to (3.30) and (3.31), and also to (3.32), Theorem 3 will be established if there exists an $\alpha > 0$ not depending on k and a natural number K such that

$$\alpha n_k \leqslant M_{n_k}(f), \qquad k \geqslant K. \tag{3.33}$$

Theorem 1 insures the existence of a $q_{n_k} \in \Pi_{n_k}$ such that

$$\| q_{n_k} \| = M_{n_k}(f). \tag{3.34}$$

In what follows $\| q_{n_k} \|$ is estimated for cases (i) and (iii) of Theorem 1. The analysis for case (ii) of Theorem 1 parallels that about to be given for case (i) and hence is omitted. Let $x^* \in I$ be such that $|q_{n_k}(x^*)| = \| q_{n_k} \|$. Define $A \subseteq E_{n_k}(f)$ by

$$A = \{x \in E_{n_k}(f) \colon \gamma_{n_k}(x)\, q_{n_k}(x) = 1\}.$$

Case (i). In this case Theorem 1 guarantees that there exist $n_k + 1$ points

$$y_0 < y_1 < \cdots < y_{n_k}$$

contained in A such that $x^* < y_0$ and $-\text{sgn}\, q_{n_k}(x^*), \gamma_{n_k}(y_0), \gamma_{n_k}(y_1), \dots, \gamma_{n_k}(y_{n_k})$ alternate in sign. Let

$$B = \{ y_0, \dots, y_{n_k} \} \subseteq A \subseteq E_{n_k} = \{x_0, \dots, x_{n_{k+1}}\}.$$

Now (2.12) and (3.31) insure that there are precisely a elements of $E_{n_k}(f) - B$. Let

$$\{z_0, z_1, \dots, z_{a-1}\} = E_{n_k}(f) - B.$$

For Case (i), $z_0 = -1 \in E_{n_k}(f) - B$. Define $p \in \Pi_{n_k+1}$ by

$$\begin{aligned} p(y_i) = q_{n_k}(y_i) = \gamma_{n_k}(y_i), \qquad i = 0, \dots, n_k, \\ p(-1) = -\text{sgn}\, q_{n_k}(x^*). \end{aligned} \tag{3.35}$$

Thus p has $n_k + 2$ sign changes. If b_{n_k+1} is the coefficient of x^{n_k+1} in p, then the argument given in [7] implies that

$$|b_{n_k+1}| \geqslant 2^{n_k}. \tag{3.36}$$

Also,

$$q_{n_k}(x) = p(x) - b_{n_k+1} \prod_{j=0}^{n_k} (x - y_j).$$

Therefore from (3.35)

$$q_{n_k}(-1) = p(-1) - b_{n_k+1} \prod_{j=0}^{n_k} (-1 - y_j)$$

$$= -\gamma_{n_k}(y_0) - b_{n_k+1} \frac{\prod_{j=1}^{m+1} (-1 - x_j)}{\prod_{j=1}^{a-1} (-1 - z_j)}.$$

Thus

$$|q_{n_k}(-1)| \geqslant |b_{n_k+1}| \frac{\prod_{j=1}^{m+1} |1 + x_j|}{\prod_{j=1}^{a-1} |1 + z_j|} - 1. \tag{3.37}$$

Applying (3.12) and (3.36) to (3.37) results in

$$|q_{n_k}(-1)| \geqslant \frac{|g(-1)|}{an_k 2^{a-2} \prod_{j=1}^{a-1} |1 + z_j|} - 1.$$

Since $\prod_{j=1}^{a-1} |1 + z_j| \leqslant 2^{a-1}$, it follows that

$$|q_{n_k}(-1)| \geqslant \frac{|g(-1)|}{an_k 2^{2a-3}} - 1.$$

Equation (3.1) now implies that

$$|q_{n_k}(-1)| \geqslant n^k(1 - t)^2 + a(1 - t^2) - 1.$$

This inequality and (3.34) imply for Case (i) that (3.33) is satisfied.

Case (iii). Verifying (3.33) for this case is slightly more complex. Let $B = \{y_0, ..., y_{i-1}, y_i, ..., y_{n_k}\} \subseteq A$ be the extreme points guaranteed by Theorem 1, where now $y_{i-1} < x^* < y_i$, for some $i = 1, ..., n_k$. Since $\gamma_{n_k}(y_{i-1}) = \gamma_{n_k}(y_i)$ and $E_{n_k}(f) = E_m(f)$, there exists a $v \in E_{n_k}(f)$ such that $y_{i-1} < v < y_i$ and $\gamma_{n_k}(v) = -\gamma_{n_k}(y_i)$. For Case (iii), define $p \in \Pi_{n_k+1}$ by

$$p(y_i) = q_{n_k}(y_i) = \gamma_{n_k}(y_i), \qquad i = 0, ..., n_k,$$
$$p(v) = \gamma_{n_k}(v).$$

Again p has $n+2$ sign changes and hence (3.36) is valid for this p. Now Theorem 1 and the definition of p imply that

$$q_{n_k}(x) = p(x) - b_{n_k+1} \prod_{j=0}^{n_k} (x - y_j). \tag{3.38}$$

As in Case (i), if $B = \{y_0, ..., y_{n_k}\}$, then $B \subseteq A \subseteq E_{n_k} = \{x_0, ..., x_{n_{k+1}}\}$, and $E_{n_k}(f) - B$ contains precisely a elements $\{z_0, ..., z_{a-1}\}$. Therefore (3.38) may be expressed as

$$q_{n_k}(x) = p(x) - \frac{b_{n_k+1} \prod_{j=0}^{m+1} (x - x_j)}{\prod_{j=0, z_j \neq v}^{a-1} (x - z_j)(x - v)}.$$

Again by utilizing (3.12) this expression can be written as

$$q_{n_k}(x) = p(x) - \frac{b_{n_k+1}(x^2 - 1)\, g(x)}{a n_k 2^m \prod_{j=0, z_j \neq v}^{a-1} (x - z_j)(x - v)}. \tag{3.39}$$

Evaluating (3.39) at $x = v$ and using (3.36) yields

$$|q_{n_k}(v)| \geqslant \frac{|(d/dx)[((x^2 - 1)\, g(x))/(x - v)]|_{x=v}}{a n_k 2^{a-1} \prod_{j=0, z_j \neq v}^{a-1} |v - z_j|} - 1.$$

Since for Case (iii), $v \neq \pm 1$, (3.19), and the observation that $\prod_{j=0, z_j \neq v}^{a-1} |x - z_j| \leqslant 2^{a-1}$ imply that

$$|q_{n_k}(v)| \geqslant \frac{n_k(1-t)^2 + a(1-t^2)}{2^{2a-2}} - 1.$$

Hence (3.33) is valid for Case (iii) also, concluding the proof of Theorem 3.

Theorem 3 provides for the precise order of $M_n(f)$ for every $f \in \hat{\mathbf{R}}$. Other efforts [5, 7] to establish the precise order of $M_n(f)$ for specific functions f have relied on $|E_n(f)| = n + 2$ for every n. Theorem 1 allows this restriction to be circumvented for rational functions in $\hat{\mathbf{R}}$.

For certain non-rational functions (1.4) provides bounds for $M_n(f)$, but precise orders for these functions have yet to be displayed.

The authors believe that the precise order concept merits further study, particularly in light of the relationships between strong unicity and Lebesgue constants [7, 8].

References

1. M. W. Bartelt and D. P. Schmidt, On strong unicity and a conjecture of Henry and Roulier,, "Approximation Theory III" (E. W. Cheney, Ed.), pp. 187–192, Academic Press, New York, 1980.

2. M. W. BARTELT AND D. P. SCHMIDT, On Poreda's Problem for Strong unicity constants, *J. Approx. Theory*, in press.
3. E. W. CHENEY, "Introduction to Approximation Theory," McGraw-Hill, New York, 1966.
4. A. K. CLINE, Lipschitz conditions on uniform approximation operators, *J. Approx. Theory* **8** (1973), 160–172.
5. M. S. HENRY AND L. R. HUFF, On the behavior of the strong unicity constant for changing dimension, *J. Approx. Theory* **27** (1979), 278–290.
6. M. S. HENRY AND J. A. ROULIER, Lipschitz and strong unicity constants for changing dimension, *J. Approx. Theory* **22** (1978), 85–94.
7. M. S. HENRY, J. J. SWETITS, AND S. E. WEINSTEIN, Orders of strong unicity constants, *J. Approx. Theory* **31** (1981), 175–187.
8. M. S. HENRY, J. J. SWETITS, AND S. E. WEINSTEIN, Lebesgue and strong unicity constants, "Approximation Theory III" (E. W. Cheney, Ed.), pp. 507–512, Academic Press, New York, 1980.
9. T. J. RIVLIN, Polynomials of best uniform approximation to certain rational functions, *Numer. Math.* **4** (1962), 345–349.
10. D. P. SCHMIDT, A characterization of strong unicity constants, "Approximation Theory III" (E. W. Cheney, Ed.), pp. 805–810, Academic Press, New York, 1980.